Theory of Liquid Sloshing in Compartmented Cylindrical Tanks Due to Bending Excitation

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Propellant sloshing may be caused by the bending of a space vehicle that exhibits low structural frequencies. Such propellant oscillations are important because there is the possibility of extreme amplitudes if the excitation (bending) frequency is in the neighborhood of one of the natural frequencies of the fuel in the container. Forces and moments exerted by the oscillating propellant on the tank are determined due to forced bending excitation for a liquid in a circular, cylindrical, ring sector tank with a free fluid surface. Special cases, such as the tank with sector and circular cross section, are obtained by limit considerations. As is expected, force and moment of the liquid increase sharply at the resonant frequencies of the propellant. The total force and moment generally are less for a given maximum bending amplitude than for a translational motion of the same magnitude. The maximum dynamic effects occur when the free fluid surface is located in the vicinity of the point of maximum bending displacement.

Nomenclature

r,φ,z	=	cylindrical coordinates
t	=	time
$\phi = \phi_0 + \psi$	=	velocity potential
ψ	=	disturbance potential
ϕ_0	=	potential of liquid without free fluid surface
$\bar{\rho}$	=	mass density of liquid
p	=	pressure of liquid
a	=	radius of outer tank wall
b	=	radius of inner tank wall
$k = b/a$	=	diameter ratio
h	=	liquid height
g	=	longitudinal acceleration (in z direction)
ω_{mn}	=	eigenfrequencies of liquid
Ω	=	forced circular frequency
$\eta = \Omega/\omega_{mn}$	=	frequency ratio
$x_0(z)$	=	amplitude of tank excitation in x direction
$\bar{x}_0$	=	maximum amplitude of bending displacement
$y_0(z)$	=	amplitude of tank excitation in y direction
F	=	fluid force
M	=	fluid moment
$\bar{z}$	=	free fluid surface displacement, measured from the undisturbed position
u_r,u_φ,w	=	flow velocity
$J_{m/2\alpha},Y_{m/2\alpha}$	=	Bessel functions of order $m/2\alpha$ of first and second kind
ξ_{mn}	=	roots of $\Delta_{m/2\alpha}(\xi) = 0$ [see Eq. (2.13)]
ϵ_{mn}	=	roots of $J_{m/2\alpha}'(\epsilon) = 0$
$2\pi\alpha \equiv \bar{\alpha}$	=	container vertex angle

Introduction

THE constantly increasing size of space vehicles introduces more new problems in modern space technology. As vehicles lengthen, their fundamental bending frequencies become lower; as their diameters become larger, the natural frequency of the propellant becomes lower. These trends restrict the choice of the control frequency value. Acute problems result from this close grouping of frequencies because of the interaction of structure, control, and propellant sloshing.

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Performance considerations necessitate a design exhibiting low structural frequencies. Therefore, the effect of elastic vibrations of the vehicle structure upon the propellant sloshing and the control system becomes more critical because of the low frequencies and the overall low damping.

Although the sloshing frequencies are even closer to the control frequency, the effect of propellant sloshing upon stability can be handled more easily than the effect of the elastic structure, provided that the phases are chosen properly. Of the many problems involved here, the one of the propellant sloshing due to bending vibrations will be treated.

In the tanks of a missile or space vehicle, sloshing will affect the performance and stability of the vehicle seriously, even causing total flight failure in some cases. Since more than 90% of the total weight of a vehicle at launch is liquid, propellant sloshing represents an area that needs special attention even in the preliminary design stage. The tendency in modern space technology is toward a continuous increase in size of space vehicles, making investigations of this kind mandatory. Therefore, for a realistic dynamic stability and control analysis, even the effect of the oscillating propellant due to bending vibrations of the structure has to be considered.

For space vehicles, which exhibit low propellant and bending frequencies, subdivision of tanks might be of importance to reduce the vibrating sloshing masses and increase the eigenfrequency of the propellant, thus moving it farther away from the control frequency. This indicates that the sloshing frequency is becoming closer to the structural frequencies. In the following, therefore, the liquid oscillations in a cylindrical tank with circular, ring sector cross sections will be treated with respect to bending oscillations. From the results, one can obtain by limit considerations the solutions of the most important cases.

Forced Bending Oscillations

The flow field of the liquid with a free fluid surface in a circular, ring sector tank with a flat bottom, due to forced bending oscillations of the tank, can be obtained from the solution of the Laplace equation $\Delta\Phi = 0$ and the appropriate linearized boundary conditions. The cross section of the tank was considered to be always of identical shape. This certainly will be enhanced by the sector walls. Furthermore, the undisturbed free fluid surface is assumed to stay in the same plane.

The bending of the walls of a tank of an elastic space vehicle can be described as a superposition of translational,

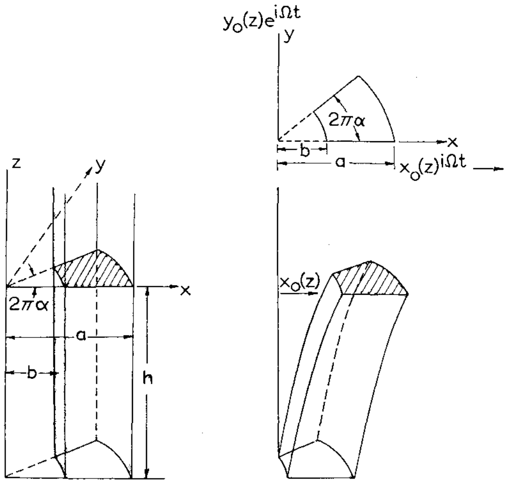

Fig. 1 Tank geometry and coordinate system

rotational, and bending oscillations with a clamped-in tank bottom. This is justified, since the theory is linearized.

The response of the liquid due to translational and pitching excitation has been treated previously.[1] The problem that remains to be solved is the response of the propellant due to bending excitation with a clamped-in tank bottom. The boundary conditions at the tank walls as well as at the free fluid surface (Fig. 1) are, for arbitrary wall oscillations (in linearized form), as follows:

$$\frac{\partial \Phi}{\partial r} = \begin{Bmatrix} i\Omega\, x_0(z) e^{i\Omega t} \cos\varphi \\ i\Omega\, y_0(z) e^{i\Omega t} \sin\varphi \end{Bmatrix} \quad \text{at the tank wall } r = a, b \qquad (2.1)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at the tank bottom } z = -h \qquad (2.2)$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial \varphi} = \begin{Bmatrix} 0 \\ i\Omega\, y_0\,(z) e^{i\Omega t} \end{Bmatrix}$$

$$\text{at the tank sector wall } \varphi = 0 \qquad (2.3)$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial \varphi} = \begin{Bmatrix} -i\Omega\, x_0(z) e^{i\Omega t} \sin 2\pi\alpha \\ i\Omega\, y_0\,(z) e^{i\Omega t} \cos 2\pi\alpha \end{Bmatrix}$$

$$\text{at the tank sector wall } \varphi = 2\pi\alpha \qquad (2.4)$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0 \quad \text{at the free fluid surface } z = 0 \qquad (2.5)$$

where the upper and lower line on the right-hand side represent excitation in the x and y directions, respectively.

Green's function was used in this treatment, since in most cases the displacement curve of the bending structure is not known analytically, and since the representation of the solution as an integral is more advantageous for the numerical evaluation on high-speed computers.

Extracting the motion of the container, the potential becomes

$$\Phi = \left[\psi + \begin{Bmatrix} i\Omega\, r\, x_0\,(z)\, \cos\varphi \\ i\Omega\, r\, y_0\,(z)\, \sin\varphi \end{Bmatrix}\right] e^{i\Omega t} \qquad (2.6)$$

One obtains for the boundary conditions of the disturbance potential

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{for } r = a, b \qquad (2.7)$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0 \quad \text{for } \varphi = 0, 2\pi\alpha \qquad (2.8)$$

$$\begin{Bmatrix} \dfrac{\partial \psi}{\partial z} \end{Bmatrix} = \begin{Bmatrix} -i\Omega x_0'(z) r \cos\varphi \\ -i\Omega y_0'(z) r \sin\varphi \end{Bmatrix} \quad \text{for } z = -h \qquad (2.9)$$

$$g\frac{\partial \psi}{\partial z} - \Omega^2 \psi = \begin{Bmatrix} i\Omega\, r \cos\varphi[\Omega^2\, x_0(0) - g\, x_0'(0)] \\ i\Omega\, r \sin\varphi[\Omega^2\, y_0(0) - g\, y_0'(0)] \end{Bmatrix}$$

$$\text{for } z = 0 \qquad (2.10)$$

Instead of the Laplace equation, the Poisson equation of the form

$$\Delta\psi = \begin{Bmatrix} -i\Omega\, x_0''(z) r \cos\varphi \\ -i\Omega\, y_0''(z) r \sin\varphi \end{Bmatrix} \qquad (2.11)$$

has to be solved with the foregoing boundary conditions.

The solution of the Poisson equation which satisfies boundary conditions (2.7) and (2.8) is of the form

$$\psi(r,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(z) \cos\left(\frac{m}{2\alpha}\varphi\right) C_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right) \equiv A(z)\cos\bar{\varphi}\, C(\rho) \qquad (2.12)$$

where

$$C_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right) \equiv C(\rho) = J_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right) Y_{m/2\alpha}'\,(\xi_{mn}) - J_{m/2\alpha}'\,(\xi_{mn})\, Y_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right)$$

The values ξ_{mn} are the positive roots of the equation

$$\Delta_{m/2\alpha} = J_{m/\alpha 2}'\,(\xi)\, Y_{m/2\alpha}'\,(k\xi) - J_{m/2\alpha}'\,(k\xi)\, Y_{m/2\alpha}'\,(\xi) = 0 \qquad (2.13)$$

The expression $k = b/a$ is the diameter ratio of the inner to the outer wall.

Here, for the sake of simplicity, the double summations and the indices m and n have been omitted. The abbreviations are

$$\bar{\varphi} = (m/2\alpha)\varphi \qquad \rho = \xi_{mn}(r/a)$$

For the determination of the unknown coefficients A_{mn}, one expands the right-hand side of the tank bottom and free surface conditions into Fourier and Bessel series. The cosine and sine can be expressed as

$$\cos\varphi = \sum_{m=0}^{\infty} a_m \cos\bar{\varphi}$$

$$\text{with } a_0 = \frac{\sin\bar{\alpha}}{\bar{\alpha}};\; a_m = \frac{2\bar{\alpha}(-1)^m \sin\bar{\alpha}}{(m^2\,\pi^2 - \bar{\alpha}^2)};\; (\bar{\alpha} = 2\pi\alpha)$$

$$\sin\varphi = \sum_{m=0}^{\infty} c_m \cos\bar{\varphi}$$

$$\text{with } c_0 = \frac{1-\cos\bar{\alpha}}{\bar{\alpha}};\; c_m = \frac{2\bar{\alpha}[(-1)^m \cos\bar{\alpha} - 1]}{(m^2\,\pi^2 - \bar{\alpha}^2)} \qquad (2.14\text{a})$$

The radius r can be expressed in the form

$$r = \sum_{n=0}^{\infty} b_{mn}\, C(\rho) \qquad (2.14\text{b})$$

$$m = 0, 1, 2 \ldots$$

in which the coefficients are (see Appendix)

$$b_{mn} = \frac{a\displaystyle\int_{k\xi_{mn}}^{\xi_{mn}} \rho^2\, C(\rho)\, d\rho}{\xi_{mn}\displaystyle\int_{k\xi_{mn}}^{\xi_{mn}} \rho\, C^2(\rho)\, d\rho} = \frac{2\,a\,N_2\,(\xi_{mn})}{[(4/\pi^2\xi_{mn}^2) - k^2C^2(k\xi_{mn})] - (m^2/4\alpha^2\xi_{mn}^2)[(4/\pi^2\xi_{mn}^2) - C^2\,(k\xi_{mn})]} \qquad (2.15)$$

With the forementioned series expansion, one obtains an infinite number of ordinary differential equations:

$$A_{mn}''(z) - \frac{\xi_{mn}^2}{a^2} A_{mn}(z) = \begin{Bmatrix} -i\,\Omega\, b_{mn}\, a_m\, x_0''(z) \\ -i\,\Omega\, b_{mn}\, c_m\, y_0''(z) \end{Bmatrix} \quad (2.16)$$

$$(m, n = 0, 1, 2 \ldots)$$

These differential equations are solved with Green's function. From the boundary conditions at the tank bottom and free fluid surface, one obtains with the series expansions [Eqs. (2.14a) and (2.14b)]

$$A_{mn}'(-h) = \begin{Bmatrix} -i\,\Omega\, b_{mn}\, a_m\, x_0'(-h) \\ -i\,\Omega\, b_{mn}\, c_m\, y_0'(-h) \end{Bmatrix}$$

$$g\, A_{mn}'(0) - \Omega^2 A_{mn}(0) = \begin{Bmatrix} [\Omega^2 x_0(0) - g x_0'(0)] i\Omega b_{mn}\, a_m \\ [\Omega^2 y_0(0) - g y_0'(0)] i\Omega b_{mn}\, c_m \end{Bmatrix}$$

A translational motion $x_0\,(-h)^{e^{i\Omega t}}$ or $y_0\,(-h)^{e^{i\Omega t}}$ is superimposed.

Taking $x_0\,(-h) = 0$ or $y_0\,(-h) = 0$, the clamped-in condition is obtained, whereas for $x_0' = 0$ or $y_0' = 0$ the translational excitation case results.

Green's function finally is

$$A_{imn} = G_i\,(z,\zeta) = \alpha_i(\zeta) e^{(\xi_{mn}/a)z} + \beta_i(\zeta) e^{-(\xi_{mn}/a)z} \quad (2.17)$$

$$i = 1 \text{ for } -h \le z \le \zeta$$

$$i = 2 \text{ for } \zeta \le z \le 0$$

At $z = \zeta$,

$$G_1\,(\zeta,\zeta) = G_2\,(\zeta,\zeta)$$

and the first derivative of Green's function has a discontinuity of unity:

$$G_1'(\zeta,\zeta) - G_2'\,(\zeta,\zeta) = -1$$

With this, the four unknown values α_1, α_2, β_1, and β_2 can be determined.

With $\eta = \Omega/\omega_{mn}$ as the ratio of exciting- to eigenfrequency and

$$\omega_{mn}^2 = (g\xi_{mn}/a) \tanh[\xi_{mn}(h/a)]$$

Green's function is

The solution of the differential equation (2.16) is then

$$A_{mn}(z) = \frac{i\,\Omega\, ab_{mn}\begin{Bmatrix} a_m \\ c_m \end{Bmatrix} \eta^2}{\xi_{mn} \cosh[\xi_{mn}(h/a)](1 - \eta^2)} \cdot \left\{\left[\sinh\left(\xi_{mn}\frac{z}{a}\right) + \frac{\xi_{mn}\, g}{a\,\Omega^2} \cosh\left(\xi_{mn}\frac{z}{a}\right)\right] \cdot \left[\begin{Bmatrix} x_0'(-h) \\ y_0'(-h) \end{Bmatrix} + \int_{-h}^{z} \begin{Bmatrix} x_0''(\zeta) \\ y_0''(\zeta) \end{Bmatrix} \cdot \cosh\left[\xi_{mn}\left(\frac{\zeta}{a} + \frac{h}{a}\right)\right] d\zeta \right] + \cosh\left[\xi_{mn}\left(\frac{z}{a} + \frac{h}{a}\right)\right] \cdot \left[\int_{-z}^{0} \begin{Bmatrix} x_0''(\zeta) \\ y_0''(\zeta) \end{Bmatrix} \left[\sinh\left(\xi_{mn}\frac{\zeta}{a}\right) + \frac{\xi_{mn} g}{a\Omega^2} \cosh\left(\xi_{mn}\frac{\xi}{a}\right)\right] d\zeta + \frac{\xi_{mn}}{a} \begin{Bmatrix} [x_0(0) - (g/\Omega^2) x_0'(0)] \\ [y_0(0) - (g/\Omega^2) y_0'(0)] \end{Bmatrix} \right]\right\} \quad (2.23)$$

and velocity potential is

$$\Phi(r,\varphi,z,t) = e^{i\Omega t}\left[\begin{Bmatrix} i\,\Omega\, x_0(z) r \cos\varphi \\ i\,\Omega\, y_0(z)\, r \sin\varphi \end{Bmatrix} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(z) C_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right) \cos\left(\frac{m}{2\alpha}\varphi\right)\right] \quad (2.24)$$

The term in front of the double summation satisfies the boundary conditions at the tank walls, whereas the terms of the double series vanish at the tank walls. The double summation, together with the term in front of it, satisfies the free surface condition, if one considers the results of Eqs. (2.14a) and (2.14b) and the eigenvalues $\omega_{mn}^2 = (g\xi_{mn}/a) \tanh[\xi_{mn}(h/a)]$. With this velocity potential, the free surface displacement, the pressure and velocity distribution, and the forces and moments of the liquid can be obtained by differentiations and integrations with respect to the time and spatial coordinates.

The pressure in a depth $(-z)$ is

$$p = -\bar{\rho}\frac{\partial\Phi}{\partial t} - g\bar{\rho}z = \bar{\rho}\Omega^2 e^{i\Omega t}\left[\begin{Bmatrix} x_0(z)\, r \cos\varphi \\ y_0(z)\, r \sin\varphi \end{Bmatrix} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(z) C_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right) \cos\left(\frac{m}{2a}\varphi\right)\right] - \bar{\rho}\, g\, z \quad (2.25)$$

$$G_1\,(z,\zeta) = -\frac{\{\sinh[(\xi_{mn}/a)\zeta] + (\xi_{mn}g/a\,\Omega^2)\cosh[\xi_{mn}(\zeta/a)]\}}{(\xi_{mn}/a)\cosh[\xi_{mn}(h/a)](1 - \eta^2)}\,\eta^2 \cosh\left[\xi_{mn}\left(\frac{z}{a} + \frac{h}{a}\right)\right] \quad \text{for } -h \le z \le \zeta \quad (2.18)$$

$$G_2\,(z,\zeta) = -\frac{\{\sinh[\xi_{mn}(z/a)] + (\xi_{mn}g/a\,\Omega^2)\cosh[\xi_{mn}(z/a)]\}}{(\xi_{mn}/a)\cosh(\xi_{mn}h/a)(1 - \eta^2)}\,\eta^2 \cosh\left[\xi_{mn}\left(\frac{\zeta}{a} + \frac{h}{a}\right)\right] \quad \text{for } \zeta \le z \le 0 \quad (2.19)$$

The solution for the homogeneous boundary conditions is then

$$A_{mn}(z) = i\Omega b_{mn}\begin{Bmatrix} a_m \\ c_m \end{Bmatrix} \frac{\{\sinh[\xi_{mn}(\zeta/a)] + (\xi_{mn}g/a\,\Omega^2)\cosh[\xi_{mn}(\zeta/a)]\}}{(\xi_{mn}/a)\cosh[(\xi_{mn}/a)h]\,(1 - \eta^2)}\,\eta^2 \cdot \int_{-h}^{z} \begin{Bmatrix} x_0''(\zeta) \\ y_0''(\zeta) \end{Bmatrix} \cosh\left[\frac{\xi_{mn}}{a}(\zeta + h)\right] d\zeta + i\Omega b_{mn}\begin{Bmatrix} a_m \\ c_m \end{Bmatrix} \frac{\cosh\{\xi_{mn}[(z/a) + (h/a)]\}}{(\xi_{mn}/a)\cosh[\xi_{mn}(h/a)](1 - \eta^2)}\,\eta^2 \int_{z}^{0} \begin{Bmatrix} x_0''(\zeta) \\ y_0''(\zeta) \end{Bmatrix} \left[\sinh\left(\xi_{mn}\frac{\zeta}{a}\right) + \frac{g\xi_{mn}}{a\Omega^2}\cosh\left(\xi_{mn}\frac{\zeta}{a}\right)\right] d\zeta \quad (2.20)$$

The solution for the inhomogeneous boundary condition

$$A_{mn}(-h) = 1 \qquad g\, A_{mn}'(0) - \Omega^2 A_{mn}(0) = 0$$

is

$$i\Omega b_{mn}\,\eta^2 \begin{Bmatrix} a_m \\ c_m \end{Bmatrix} \frac{\{(\xi_{mn}\, g/a\Omega^2)\cosh[\xi_{mn}(z/a)] + \sinh[\xi_{mn}(z/a)]\}}{(\xi_{mn}/a)\cosh[\xi_{mn}(h/a)](1 - \eta^2)} \quad (2.21)$$

and the one for the boundary conditions

$$A_{mn}'(-h) = 0 \qquad g\, A_{mn}'(0) - \Omega^2 A_{mn}(0) = 1$$

is

$$i\Omega b_{mn}\,\eta^2 \begin{Bmatrix} {}_m[x_0(0) - (g/\Omega^2) x_0'(0)] \\ c_m[y_0(0) - (g/\Omega^2) y_0'(0)] \end{Bmatrix} \cdot \frac{\cosh\{\xi_{mn}[(z/a) + (h/a)]\}}{\cosh[\xi_{mn}(h/a)](1 - \eta^2)} \quad (2.22)$$

Table 1 Sector tank

Free surface displacement:

$$\bar{z} = \frac{\Omega^2 e^{i\Omega t}}{g}\left[\begin{Bmatrix} r x_0(0)\cos\varphi \\ r y_0(0)\sin\varphi \end{Bmatrix} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(0) J_{m/2\alpha}\left(\epsilon_{mn}\frac{r}{a}\right)\cdot \cos\left(\frac{m}{2\alpha}\varphi\right)\right]$$

Fluid force:

$$F_x = m\Omega^2 e^{i\Omega t}\left[\frac{1}{h}\int_{-h}^{0}\begin{Bmatrix} x_0(z) \\ 0 \end{Bmatrix} dz + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}\sin 2\pi\alpha}{i\Omega\pi a\, h\, \alpha}\cdot\left[\frac{4\alpha^2}{(m^2-4\alpha^2)}\cdot J_{m/2\alpha}(\epsilon_{mn}) + L_0(\epsilon_{mn})\right]\cdot\int_{-h}^{0} A_{mn}^{\left(\substack{x\\y}\right)}(z)\,dz\right]$$

$$F_y = m\Omega^2 e^{i\Omega t}\left[\frac{1}{h}\int_{-h}^{0}\begin{Bmatrix} 0 \\ y_0(z) \end{Bmatrix} dz + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{[(-1)^m\cos 2\pi\alpha - 1]}{i\Omega\alpha a}\cdot\frac{1}{h}\int_{-h}^{0} A_{mn}^{\left(\substack{x\\y}\right)}(z)\,dz\cdot\left[\frac{4\alpha^2}{(m^2-4\alpha^2)} J_{m/2\alpha}(\epsilon_{mn}) + L_0(\epsilon_{mn})\right]\right]$$

Fluid moment:

$$M_y = m\Omega^2 e^{i\Omega t}\left[\begin{Bmatrix} \dfrac{1}{h}\displaystyle\int_{-h}^{0}\left(\frac{h}{2}+z\right)x_0(z)\,dz + \dfrac{x_0(-h)a^2}{4h}\left(\dfrac{1+\sin 2\pi\alpha\cos 2\pi\alpha}{2\pi\alpha}\right) \\ 0 + \dfrac{y_0(-h)a^2}{4h}\cdot\dfrac{\sin^2 2\pi\alpha}{2\pi\alpha} \end{Bmatrix} + \right.$$

$$\left.\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}\sin 2\pi\alpha}{i\,\Omega\pi\alpha\, a\, h}\left[\frac{4\alpha^2 a^2 L_2(\epsilon_{mn})}{(m^2-4\alpha^2)} A_{mn}^{\left(\substack{x\\y}\right)}(-h) + \left(\frac{4\alpha^2}{(m^2-4\alpha^2)} J_{m/2\alpha}(\epsilon_{mn}) + L_0(\epsilon_{mn})\right)\cdot\int_{-h}^{0}\left(\frac{h}{2}+z\right)A_{mn}^{\left(\substack{x\\y}\right)}(z)\,dz\right]\right] + mg\frac{a}{3}\frac{\sin 2\pi\alpha}{\pi\alpha}$$

$$M_x = -m\Omega^2 e^{i\Omega t}\left[\begin{Bmatrix} \dfrac{x_0(-h)\,a^2}{4h}\cdot\dfrac{\sin^2 2\pi\alpha}{2\pi\alpha} + 0 \\ \dfrac{y_0(-h)a^2}{4h}\left(1 - \dfrac{\sin 2\pi\alpha\cos 2\pi\alpha}{2\pi\alpha}\right) + \dfrac{1}{h}\displaystyle\int_{-h}^{0}\left(\frac{h}{2}+z\right)y_0(z)\,dz \end{Bmatrix} + \right.$$

$$\left.\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{[(-1)^m\cos 2\pi\alpha - 1]}{i\Omega\pi\alpha\, ah}\cdot\left[\frac{4\alpha^2 a^2}{(m^2-4\alpha^2)} A_{mn}^{\left(\substack{x\\y}\right)}(-h)L_2(\epsilon_{mn}) + \left(\frac{4\alpha^2}{(m^2-4\alpha^2)} J_{m/2\alpha}(\epsilon_{mn}) + L_0(\epsilon_{mn})\right)\cdot\int_{-h}^{0}\left(\frac{h}{2}+z\right)\cdot A_{mn}^{\left(\substack{x\\y}\right)}(z)\,dz\right]\right] - mg\frac{a}{3}\frac{[1-\cos 2\pi\alpha]}{\pi\alpha}$$

At the outer tank wall $r = a$, the value of $C_{m/2\alpha}[\xi_{mn}(r/a)]$ is $2/\pi\xi_{mn}$, whereas at the inner tank wall $r = b$, it is $C_{m/2\alpha}(k\,\xi_{mn})$.

At the sector walls $\varphi = 0$ and $\varphi = 2\pi\alpha$, the cosine function has the value 1 and $(-1)^m$, respectively. The pressure distribution at the tank bottom results from Eq. (2.25) by setting $z = -h$.

From the pressure distribution, the liquid forces and moments are determined by integrating the appropriate components. In the x direction, the force is

$$F_x = \int^{2\pi\alpha}\int_{-h}^{0}(ap_a - bp_b)\cos\varphi\, d\,\varphi\, d\,z - \int_b^a\int_{-h}^{0} p_{\varphi=2\pi\alpha}\sin 2\pi\alpha\, dr\, dz$$

The first integral represents the contribution of the pressure distribution from the circular walls, whereas the remaining integral is due to the pressure distribution at the sector walls. The force component in the y direction can be obtained in a similar way. The liquid moment is

$$M_y = \int_0^{2\pi\alpha}\int_{-h}^{0}(ap_a - b\,p_b)\left(\frac{h}{2}+z\right)\cos\varphi\, d\varphi\, dz + \int_0^{2\pi\alpha}\int_b^a p_c\, r^2\cos\varphi\, d\varphi\, dr - \int_b^a\int_{-h}^{0} p_{\varphi=2\pi\alpha}\cdot \sin 2\pi\alpha\left(\frac{h}{2}+z\right)dr\, dz$$

where M_y is the moment about the parallel axis to the y axis through the point $(0, 0, -h/2)$. The first integral represents the contribution of the pressure distribution from the circular walls. The second integral is the contribution of the pressure at the tank bottom, whereas the remaining integral easily can be identified as the contribution of the pressure distribution at the tank sector wall.

Special Cases

Tanks with circular cross sections are at present the most frequent containers. Tendencies in space technology, however, point toward the intersection of the tank by radial walls. Subdivision of a cylindrical tank into four quarter-compartments is a possibility to reduce the dynamic influence of the propellant sloshing upon the stability of the space vehicle. Concentric containers could be beneficial, if one could choose their diameter ratio in such a way that the phasing of the sloshing mass of the propellant inside and outside is such that their combined effect cancels. The theory of liquid motion due to bending tank walls already was presented for annular cylindrical tanks.[2] The following will be restricted to the sector and circular tank. The main results are given in Table 1.

A. Sector Tank

If the diameter ratio $k = b/a$ tends to zero, then the results of Eq. (2.2) represent the motion of a liquid with free fluid surface in a tank with circular sector cross section. The zeros of the determinant $\Delta_{m/2\alpha}(\xi) = 0$ now are obtained from the equation $J_{m/2\alpha}'(\epsilon) = 0$ and are noted by ϵ_{mn}. Furthermore, the function $C(\rho)$ is substituted by $J(\rho) \equiv J_{m/2\alpha}[\epsilon_{mn}(r/a)]$. The integration constants b_{mn} transform into

$$\bar{b}_{mn} = \frac{a\displaystyle\int_0^{\epsilon_{mn}}\rho^2 J(\rho)\,d\rho}{\epsilon_{mn}\displaystyle\int_0^{\epsilon_{mn}}\rho J^2(\rho)\,d\rho} \qquad m, n = 0, 1, 2, \ldots$$

which is

$$b_{mn} = 2a \frac{\dfrac{\Gamma(m/4\alpha + \frac{3}{2})}{\Gamma(m/4\alpha - \frac{1}{2})} \sum_{\mu=0}^{\infty} \dfrac{(m/2\alpha + 2\mu + 1)\Gamma(m/4\alpha + \mu - \frac{1}{2})}{\Gamma(m/4\alpha + \mu + \frac{5}{2})} J_{m/2\alpha+2\mu+1}(\epsilon_{mn})}{\epsilon_{mn}(1 - m^2/4\alpha^2 \epsilon_{mn}^2) J_{m/2\alpha}^2(\epsilon_{mn})} \quad (3.1)$$

In the force components, one has to omit the singular solution at $r = 0$. Therefore, the value $[2/\pi\xi_{mn} - kC_{m/2\alpha}(k\xi_{mn})]$ has to be substituted by $J_{m/2\alpha}(\epsilon_{mn})$. The velocity potential, the free fluid surface displacement, and the force and moment components are represented in Table 1.

B. Circular Cylindrical Tank

For a container with circular cross section, $\alpha = 1$. This represents a container with a side wall in the $\varphi = 0$ plane from $r = 0$ to $r = a$. The values a_m are then

$$a_0 = a_m = 0 \qquad a_2 = \lim_{\alpha \to 1} \left\{ -\frac{\alpha \sin 2\pi\alpha}{\pi(1 - \alpha^2)} \right\} = 1$$

If one chooses an excitation in the x direction, the side wall does not disturb the flow field. The expression b_{2n} is obtained from (3.1). Because of the singularity of the gamma function at the argument zero, only one term occurs. With the recurrence formula of the Bessel functions $x\ J_v'(x) - v\ J_v\ (x) = -x\ J_{v+1}\ (x)$, one obtains [because of ϵ_n being the zeros of $J_1'(\epsilon_n) = 0$] for $J_2\ (\epsilon_n) = \epsilon_n\ J_1\ (\epsilon_n)$. Therefore

$$b_{2n} = \frac{2a}{(\epsilon_n^2 - 1)J_1(\epsilon_n)}$$

The velocity potential therefore is due to bending excitation in the x direction:

$$\Phi(r,\varphi,z,t) = i\Omega e^{i\Omega t} a \cos\varphi \left\{ \frac{r}{a} x_0(z) + 2\sum_{n=1}^{\infty} \frac{J_1[\epsilon_n(r/a)]\eta^2}{(\epsilon_n^2 - 1)J_1(\epsilon_n)(\epsilon_n/a)\cosh[\epsilon_n(h/a)](1 - \eta^2)} \cdot \left[\left(\sinh\left(\epsilon_n \frac{z}{a}\right) + \frac{\epsilon_n g}{a\Omega^2} \cosh\left(\epsilon_n \frac{z}{a}\right) \right) \cdot \left(x_0'(-h) + \int_{-h}^{z} x_0''(\zeta) \cosh\left[\frac{\epsilon_n}{a}(\zeta + h)\right] d\zeta \right) + \cosh\left[\frac{\epsilon_n}{a}(z + h)\right] \cdot \left(\int_z^0 x_0''(\zeta) \left[\sinh\left(\epsilon_n \frac{\zeta}{a}\right) + \frac{\epsilon_n g}{a\Omega^2} \cdot \cosh\left(\frac{\epsilon_n \zeta}{a}\right) \right] d\zeta + \frac{\epsilon_n}{a}\left[x_0(0) - \frac{g}{\Omega^2} x_0'(0) \right] \right) \right] \right\} \quad (3.2)$$

The force and moment of the liquid are represented in Table 2.

In the numerical evaluation, a bending displacement is considered which has the form (Fig. 2)

$$x_0(z) = -\frac{4a}{250}\left[\left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right) - 6\right]$$

The maximum amplitude of this displacement is $\bar{x}_0/a = \frac{1}{10}$. The result for this particular case also could be obtained from Ref. 7.

Figure 3 exhibits the fluid force vs fluid height ratio h/a for various exciting frequency ratios $\eta_i = \Omega/\omega_i$. The fluid height has, of course, considerable influence, since the exciting amplitude is changing along the z axis. The moment of the liquid (Fig. 3) exhibits similar behavior.

The total force and moment generally are less for a given maximum bending amplitude than for a translational motion of the same magnitude. The maximum dynamic effects occur when the free fluid surface is located in the vicinity of the point of maximum bending displacement. With increasing fluid height, the contribution of the sloshing mass decreases, whereas the inertial force increases. With increasing exciting frequency, this inertial force becomes larger. Close to liquid resonance, therefore, the inertial effect is not strong enough to overcome the decrease of the effect of the sloshing fluids due to the diminished local exciting amplitude with increasing fluid height. Pressure and velocity distribution easily can be obtained.[3]

Appendix

A. Roots of Certain Bessel Functions

For the previous results, the roots of $\Delta_{m/2\alpha}(\xi) = 0$ have to be determined for $m = 0, 1, 2, \ldots$ and arbitrary $0 \le k < 1$. For most of these roots, McMahon represented asymptotic expansions.[4] The smallest root, however, was not known until Buchholz pointed out its existence.[5] Kirkham[6] gave the roots of the foregoing equation in a graphical way for $m/2\alpha = 0, 1, 2, 3, 4$.

B. Representation of a Function in Bessel-Fourier Series

A function $f\ (r)$, which is piecewise regular in the interval $b \le r \le a$, satisfies the Dirichlet condition and can be expanded into a Bessel-Fourier series of the form

$$f(r) = \sum_{n=0}^{\infty} b_{mn}^{(f)} C_{m/2\alpha}(\lambda_{mn} r) \qquad (m = 0, 1, 2, \ldots)$$

The unknown coefficients of the expansion will be determined by multiplying both sides of the equation with $rC_{m/2\alpha}(\lambda_{mn} r)$ and integrating from $r = b$ to $r = a$. Here λ_{mp} and λ_{mn} are different roots of the determinant $\Delta_{m/2\alpha} = 0$. With the integral of Lommel and the roots of $\Delta_{m/2\alpha} = 0$, one obtains for the coefficients for which $\lambda_{mn} \ne \lambda_{mp}$

$$b_{mn}^{(f)} = \frac{\int_b^a r f(r)\, C_{m/2\alpha}(\lambda_{mn} r)\, dr}{\int_b^a r\, C_{m/2\alpha}^2(\lambda_{mn} r)\, dr}$$

One thus obtains in the interval $b \le r \le a$

$$\int_b^a r\, C_{m/2\alpha}^2(\lambda_{mn} r)\, dr = \left\{ \frac{a^2}{2}\left[C_{m/2\alpha}^2(\lambda_{mn} a)\left(1 - \frac{m^2}{4\alpha^2\lambda_{mn}^2 a^2}\right) + C_{m/2\alpha}'^2(\lambda_{mn} a) \right] - \frac{b^2}{2}\left[C_{m/2\alpha}^2(\lambda_{mn} b)\left(1 - \frac{m^2}{4\alpha^2\lambda_{mn}^2 b^2}\right) + C_{m/2\alpha}'^2(\lambda_{mn} b) \right] \right\}$$

which is due to the boundary conditions

$$\int_b^a r\, C_{m/2\alpha}^2\left(\xi_{mn}\frac{r}{a}\right) dr = \frac{a^2}{2\xi_{mn}^2}\left[\frac{4}{\pi^2\xi_{mn}^2}\left(\xi_{mn}^2 - \frac{m^2}{4\alpha^2}\right) - C_{m/2\alpha}^2(k\xi_{mn})\left(k^2\xi_{mn}^2 - \frac{m^2}{4\alpha^2}\right) \right]$$

Here, $C_{m/2\alpha}(\xi_{mn}) = 2/\pi\xi_{mn}$ is the Wronskian determinant.

The coefficient $b_{mn}^{(f)}$ of the Bessel-Fourier expansion can be determined from

$$b_{mn}^{(f)} = \frac{2\xi_{mn}^2 \int_b^a r\, f(r)\, C_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right) dr}{a^2\{(4/\pi^2\xi_{mn}^2)[\xi_{mn}^2 - (m^2/4\alpha^2)] - C^2(k\xi_{mn})[k^2\xi_{mn}^2 - (m^2/4\alpha^2)]\}}$$

The problem that remains is the solution of the

$$\int_b^a r\, f(r)\, C_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right) dr$$

Most of the integrals in the previous treatment are of the form

$$\int z^\kappa C_\nu(z) dz$$

These can be obtained with the help of the Lommel function $S_{\kappa\nu}(z)$ or by integration of the series expansion of the integrals:

$$\int z^\kappa C_{m/2\alpha}(z)\, dz = Y_{m/2\alpha}'(\xi_{mn}) \int z^\kappa J_{m/2\alpha}(z)\, dz - J_{m/2\alpha}'(\xi_{mn}) \int z^\kappa Y_{m/2\alpha}(z)\, dz$$

Integrating the first integral term by term and collecting terms of $J_{\nu+2\mu+1}$, one obtains[8]

$$\int z^\kappa J_\nu(z)\, dz = \frac{z^\kappa \Gamma[(\kappa+\nu+1)/2]}{\Gamma[(\nu-\kappa+1)/2]} \sum_{\mu=0}^{\infty} \frac{(\nu+2\mu+1)\Gamma\{[(\nu+\kappa+1)/2]+\mu\}}{\Gamma\{[(\nu+\kappa+3)/2]+\mu\}} J_\nu + 2\mu + 1$$

where $Re\,(\kappa+\nu+1) > 0$ if one integrates from $z = 0$.

The second integral is obtained by termwise integration of the series expansion of the Bessel function of the second kind. It is (for $m/2\alpha$ integer)

$$\int z^\kappa Y_{m/2\alpha}(z)\, dz = -\frac{z^{\kappa+1}}{\pi} \sum_{\mu=0}^{m/2\alpha-1} \frac{[(m/2\alpha)-\mu-1]!(z/2)^{2\mu-(m/2\alpha)}}{\mu![\kappa+2\mu-(m/2\alpha)+1]} +$$
$$\frac{2z^{\kappa+1}}{\pi}\sum_{\mu=0}^{\infty} \frac{(-1)\,\mu\,(z/2)^{(m/2\alpha)+2\mu}}{\mu![(m/2\alpha)+\mu]![(m/2\alpha)+\kappa+2\mu+1]^2} \cdot \left\{\ln\frac{z}{2} - \frac{1}{2}\psi(\mu+1) - \frac{1}{2}\psi\left(\mu+\frac{m}{2\alpha}+1\right)\right\} -$$
$$\frac{2z^{\kappa+1}}{\pi}\sum_{\mu=0}^{\infty} \frac{(-1)^\mu (z/2)^{(m/2\alpha)+2\mu}}{\mu![(m/2\alpha)+\mu]![(m/2\alpha)+\kappa+2\mu+1]^2}$$

Table 2 Circular cylinder tank

Fluid force:

$$F_x = m\Omega^2 e^{i\Omega t}\left\{\frac{1}{h}\int_{-h}^0 x_0(z)dz + 2\sum_{n=1}^{\infty}\frac{\eta^2}{(\epsilon_n^2-1)[\epsilon_n(h/a)]\cosh[\epsilon_n(h/a)](1-\eta^2)} \cdot \left[x_0'(-h)\left(\frac{a}{\epsilon_n}\left[1-\cosh\frac{\epsilon_n h}{a}\right] + \right.\right.\right.$$
$$\left.\frac{g}{\Omega^2}\sinh\left(\epsilon_n\frac{h}{a}\right)\right) + \int_{-h}^0\left[\left(\sinh\frac{\epsilon_n z}{a} + \frac{\epsilon_n g}{a\Omega^2}\cosh\left(\frac{\epsilon_n z}{a}\right)\right)\cdot\int_{-h}^z x_0''(\zeta)\cosh\left(\frac{\epsilon_n}{a}(\zeta+h)\right)d\zeta\right]dz + \left((x_0(0) - \frac{g}{\Omega^2}x_0'(0)\right)\cdot$$
$$\left.\left.\sinh\left(\frac{\epsilon_n h}{a}\right) + \int_{-h}^0\cosh\left[\frac{\epsilon_n}{a}(z+h)\right]\cdot\int_z^0 x_0''(\zeta)\left[\sinh\left(\frac{\epsilon_n}{a}\zeta\right) + \frac{\epsilon_n g}{a\Omega^2}\cosh\left(\frac{\epsilon_n\zeta}{a}\right)\right]d\zeta dz\right]\right\}$$

$F_y = 0$

Fluid moment:

$$M_y = ma^2\,\Omega^2 e^{i\Omega t}\left\{\frac{x_0(-h)}{4h} + \frac{1}{a^2}\int_{-h}^0\left(\frac{1}{2}+\frac{z}{h}\right)x_0(z)\,dz + 2\sum_{n=1}^{\infty}\frac{\eta^2}{(\epsilon_n^2-1)[\epsilon_n(h/a)]\cosh[\epsilon_n(h/a)](1-\eta^2)a^2}\cdot\right.$$
$$\left[x_0'(-h)\left\{\left(\frac{ha}{2\epsilon_n} - \frac{ag}{\epsilon_n\Omega^2}\right) + \cosh\left(\epsilon_n\frac{h}{a}\right)\left(\frac{ah}{2\epsilon_n} + \frac{ag}{\epsilon_n\Omega^2}\right) - \sinh\left(\epsilon_n\frac{h}{a}\right)\left(\frac{2a^2}{\epsilon_n^2} + \frac{gh}{2\Omega^2}\right)\right\} + \right.$$
$$\frac{a^2}{\epsilon_n^2}\int_{-h}^0 x_0''(\zeta)\sinh\left(\frac{\epsilon_n\zeta}{a}\right)d\zeta + \frac{ag}{\epsilon_n}\int_{-h}^z x_0''(\zeta)\cosh\left(\frac{\epsilon_n}{a}\zeta\right)d\zeta + \frac{h}{2}\int_{-h}^0\left[\sinh\left(\frac{\epsilon_n}{a}z\right)\cdot\int_z^0 x_0''(\zeta)\cosh\left(\frac{\epsilon_n}{a}(\zeta+h)\right)d\zeta\right]dz +$$
$$\frac{\epsilon_n gh}{2a\Omega^2}\int_{-h}^0\left[\cosh\left(\frac{\epsilon_n}{a}z\right)\cdot\int_{-h}^z x_0''(\zeta)\cosh\left(\frac{\epsilon_n}{a}(\zeta+h)\right)d\zeta\right]dz + \frac{h}{2}\int_{-h}^0\left[\cosh\left(\frac{\epsilon_n}{a}(z+h)\right)\cdot\int_{-z}^0 x_0''(\zeta)\sinh\left(\frac{\epsilon_n}{a}\zeta\right)d\zeta\right]dz +$$
$$\frac{\epsilon_n hg}{2a\Omega^2}\int_{-h}^0\left[\cosh\left(\frac{\epsilon_n}{a}(z+h)\right)\cdot\int_z^0 x_0''(\zeta)\cosh\left(\frac{\epsilon_n}{a}\zeta\right)d\zeta\right]dz + \int_{-h}^0\left[z\sinh\left(\frac{\epsilon_n}{a}z\right)\cdot\int_{-h}^z x_0''(\zeta)\cosh\left(\frac{\epsilon_n}{a}(\zeta+h)\right)d\zeta\right]dz +$$
$$\frac{\epsilon_n g}{a\Omega^2}\int_{-h}^0\left[z\cosh\left(\frac{\epsilon_n z}{a}\right)\cdot\int_{-h}^z x_0''(\zeta)\cosh\left(\frac{\epsilon_n}{a}(\zeta+h)\right)d\zeta\right]dz + \int_{-h}^0\left[z\cosh\left(\frac{\epsilon_n}{a}(z+h)\right)\cdot\int_{-h}^0 x_0''(\zeta)\sinh\left(\frac{\epsilon_n}{a}\zeta\right)d\zeta\right]dz +$$
$$\left.\left.\frac{\epsilon_n g}{a\Omega^2}\int_{-h}^0\left[z\cosh\left(\frac{\epsilon_n}{a}(z+h)\right)\cdot\int_z^0 x_0''(\zeta)\cosh\left(\frac{\epsilon_n}{a}\zeta\right)d\zeta\right]dz - \left(x_0(0) - \frac{gx_0'(0)}{\Omega^2}\right)\cdot\left(\frac{\cosh[(\epsilon_n/a)h]}{\epsilon_n/a} - \frac{2a}{\epsilon_n} - \frac{h}{2}\sinh\left(\frac{\epsilon_n}{a}h\right)\right)\right]\right\}$$

$M_x = 0$

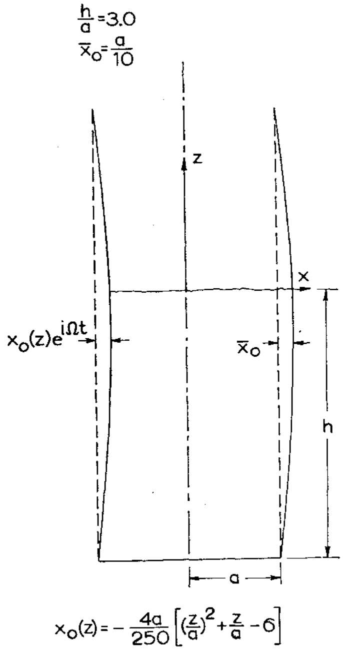

Fig. 2 Lateral bending of container

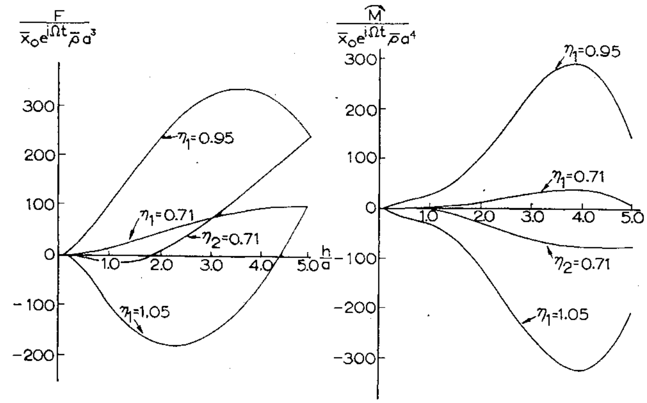

Fig. 3 Liquid force and moment vs fluid height

where $\psi(z)$ represents the logarithmic derivative of the gamma function:

$$\psi(z) = \frac{d\,[\ln\Gamma(z)]}{dz} = -\gamma + (z-1)\sum_{\lambda=0}^{\infty}\frac{1}{(\gamma+1)(z-\lambda)}$$

and γ is the Euler constant. With these results, one obtains the integral mentioned in the text:

$$\int_b^a r^2\, C_{m/2\alpha}\left(\xi_{mn}\frac{r}{a}\right) dr = a^3 N_2(\xi_{mn})$$

It may be mentioned here that some of the integrals in which μ is $1-\nu$ or $\nu+1$ can be obtained from the recursion formulas:

$$\int z^{1-\nu} C_\nu(z)\, dz = -z^{1-\nu} C_{\nu-1}(z) = -z^{1-\nu} C_\nu'(z) - z^{-\nu}\nu\, C_\nu(z)$$

$$\int z^{\nu+1} C_\nu(z)\, dz = -z^{\nu+1} C_{\nu+1}(z) = \nu z^\nu C_\nu(z) - z^{\nu+1} C_\nu'(z)$$

C. Limit Considerations for $k \to 0$

The previous results can be applied for cylindrical tanks with circular cross section by letting $k \to 0$. The zeros of the determinant Δ_ν approach for $k \to 0$ the value ϵ_{mn} for $J_{m/2n}' = 0$. This is because

$$J_\nu(x) \approx \frac{x_\nu}{2^\nu\Gamma(\nu+1)}$$

for small x and

$$Y_\nu(x) \approx \frac{-2^\nu\,\Gamma(\nu)}{\pi x^\nu}$$

for $\nu > 0$ and small x. Instead of the value $C_{m/2n}[\xi_{mn}(r/a)]$ in a ring sector tank, the values of $J_{m/2n}[\epsilon_{mn}(r/a)]$ have to be taken for a container of circular sector cross section.

The values L_0, L_2 for the sector tank are

$$L_0(\epsilon_{mn}) = \frac{2}{\epsilon_{mn}}\sum_{\mu=0}^{\infty} J_{2\mu+(m/2\alpha)+1}(\epsilon_{mn}) \qquad \left(Re\,\frac{m}{2\alpha} > -1\right)$$

$$L_2(\epsilon_{mn}) = \frac{\Gamma[(m/4\alpha)+\frac{3}{2}]}{\epsilon_{mn}\Gamma[(m/4\alpha)-\frac{1}{2}]}\sum_{\mu=0}^{\infty}\frac{[(m/2\alpha)+2\mu+1]\,\Gamma[(m/2\alpha)+\mu-\frac{1}{2}]}{\Gamma[(m/4\alpha)+\mu+\frac{5}{2}]}\cdot J_{2\mu+(m/2\alpha)+1}(\epsilon_{mn})$$

The other values can be obtained in a similar way.

References

[1] Bauer, H. F., "Treibstoffschwingungen in Raketenbehältern und ihr Einfluss auf die Gesamtstabilität," Z. Luftfahrtforsch. (submitted for publication).

[2] Bauer, H. F., "Theory of fluid oscillations in a circular cylindrical ring tank partially filled with liquid," NASA TN D-557 (December 1960).

[3] Bauer, H. F., "Fluid oscillations in a circular cylindrical tank due to bending of tank walls," Army Ballistic Missile Agency, DA TR-3-58 (April 18, 1958).

[4] McMahon, J., "On the roots of the Bessel and certain related functions," Ann. Math. **9**, 23–30 (1894).

[5] Buchholz, H., "Besondere Reihenentwicklungen für eine häufig vorkommende Zweireihige Determinante mit Zylinderfunctionen und ihre Nullstellen," Z. Angew. Math. Mech. **29**, no. 11/12, 356–367 (1949).

[6] Kirkham, D., "Graphs and formulas of zeros of cross product Bessel-functions," J. Math. Phys. **36**, 371–377 (1958).

[7] Baron, M. L. and Skalak, R., "Free vibrations of fluid-filled cylindrical shells," Proc. Am. Soc. Civil Engrs., J. Eng. Mech. Div. **88**, no. EM3, 17–43 (June 1962).

[8] Luke, Y., *Integrals of Bessel Functions* (McGraw-Hill Book Co. Inc., New York, 1962), Chap. 2.